International Journal of Bifurcation and Chaos, Vol. 21, No. 8 (2011) 2125–2145

DOI: 10.1142/S0218127411029677

KAM TORI AND ABSENCE OF DIFFUSION OF A WAVE-PACKET IN THE 1D RANDOM DNLS MODEL

S. AUBRY
Max Planck Institute für Physik Komplexer Systeme,
Nöthnitzer Str. 38, D-01187 Dresden, Germany
and
Laboratoire León Brillouin CEA Saclay,
91191 Gif-sur-Yvette, France

Received December 23, 2010; Revised February 17, 2011

When nonlinearity is added to an infinite system with purely discrete linear spectrum, Anderson modes become coupled with one another by terms of higher order than linear, allowing energy exchange between them. It is generally believed, on the basis of numerical simulations in such systems, that any initial wave-packet with finite energy spreads down chaotically to zero amplitude with second moment diverging as a power law of time, slower than standard diffusion (subdiffusion). We present results which suggest that the interpretation of spreading cannot be described as initially believed and that new questions arise and still remain opened. We show that an initially localized wave-packet with finite norm may generate two kinds of trajectories both obtained with nonvanishing probability.

The first kind consists of KAM trajectories which are recurrent and do not spread. Empirical investigations suggest that KAM theory may still hold in infinite systems under two conditions: (1) the linearized spectrum is purely discrete, (2) the considered solutions are square summable and not too large in amplitude. We check numerically that in appropriate regions of the parameter space, indeed many initial conditions can be found with finite probability that generate (nonspreading) infinite dimension tori (almost periodic solutions) in a fat Cantor set in (projected) phase space.

The second kind consists of trajectories which look initially chaotic and often spread over long times. We first rigorously prove that initial chaos does not necessarily imply complete spreading e.g. for large norm initial wave-packet. Otherwise, in some modified models, no spreading at all is proven to be possible, despite the presence of initial chaos in contradiction with early beliefs. The nature of the limit state is still unknown.

However, we attempt to present empirical arguments suggesting that if a trajectory starts chaotically spreading, there will necessarily exist (generally large) critical spreading distances that depend on the disorder realization where the trajectory will be sticking to a dense set of KAM tori. This effect should induce drastic slowing down of the spreading which could be viewed as "inverse Arnold diffusion" since the trajectory approaches KAM tori regions instead of leaving them. We suggest that this effect should self-organize the chaotic behavior and that at long time, the wave-packet might not be spread down to zero, but could have a limit profile with marginal chaos (with singular continuous spectrum), despite a long spatial tail. Further analytical and numerical investigations are required.

Keywords: KAM tori; DNLS models; infinite disordered systems.

1. Introduction

The long time behavior of a wave-packet which is initially localized in an infinite discrete system is well understood in linear media. In those integrable systems, any motion decomposes into a linear combination of eigenmodes. Thus, the properties of the linear spectrum are sufficient to determine the long time behavior of the wave-packet. If the system is spatially periodic, its linear spectrum is absolutely continuous, with a continuum of extended eigenmodes which are plane waves and not square summable. These plane waves carry energy at infinity, so that any initially localized wave-packet spreads. Spreading is defined in the sense that the maximum amplitude of the wave-packet goes uniformly to zero at infinite time.

This problem becomes much more complex for nonlinear systems, where the general solution cannot be described as a linear combination of periodic eigenmodes. Those systems are generally nonintegrable except exceptional well-known systems such as the Toda lattice or the Ablowitz–Ladik model. Though nonlinear finite size systems may exhibit chaotic trajectories characterized by a high sensitivity to initial conditions, in infinite-sized systems chaos may also manifest during rather long transient time but may not survive at infinite time. There may exist a kind of self-organization of the dynamical motion which manifest in the appearance of a regular attractor. Indeed in spatially periodic systems, it has been discovered numerically by [Sievers & Takeno, 1988] that in certain nonlinear spatially periodic systems, an initially localized wave-packet does not spread completely. Though a part of the initial energy does spread as it would in a purely linear system, the rest of the initial energy remains trapped forever as a Discrete Breather (or Intrinsic Localized Mode) (DB) which is an exact time periodic solution of the nonlinear system. Such kind of solutions do not exist in spatially periodic linear systems. It has been proved that such solutions do exist generically in large classes of nonlinear models [MacKay & Aubry, 1994; Aubry *et al.*, 2001]. Abundant literature has been devoted to DBs and their properties [Aubry, 1997, 2006; Flach & Gorbach, 2008].

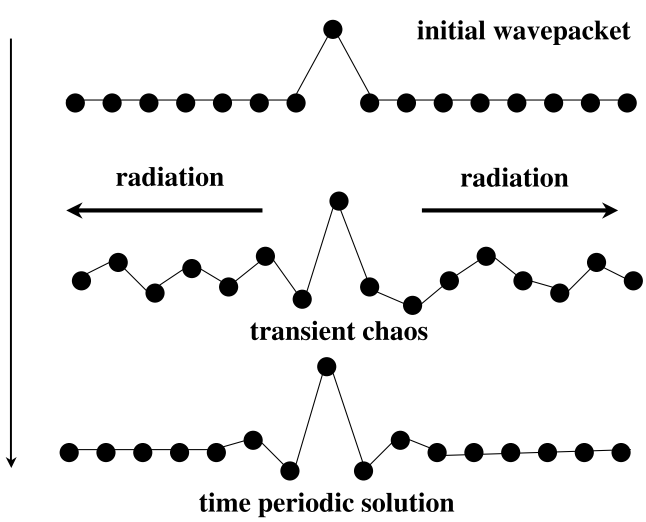


Fig. 1. Scheme of the evolution through time of an initially localized wave-packet in spatially periodic nonlinear systems which do not completely spread. In [Sievers & Takeno, 1988], a single site wave-packet radiates at infinity a part of its energy during a rather chaotic transient, and finally converges to a stationary time periodic solution which is a Discrete Breather.

The empirical argument for the existence of DBs is that in nonlinear systems the frequency of a localized periodic oscillation naturally depends on its amplitude. For a spatially periodic system, the linearized spectrum obtained for small amplitude oscillations, is absolutely continuous and consists of frequency bands with gaps. If the frequency of this oscillation and its harmonics do not belong to any band, no linear radiation at infinity is possible. Then we could expect this localized vibration to be stable and to survive forever as a DB. Despite this argument taking into account only the radiation property of the linearized system, it is supported by rigorous proofs for the existence of (linearly stable) DBs [MacKay & Aubry, 1994] which also take into account the full nonlinearity of the model. However, there is still no rigorous proof that such solutions could be an attractor for an initial wave-packet with appropriate shapes and amplitudes. Nevertheless, numerical evidences and some theoretical investigations [Johansson & Aubry, 2000] suggest that if the initial wave-packet is not too far from a DB solution, the asymptotic state is a DB.

In disordered linear systems with strong enough disorder in 3D or at any disorder in 1 and 2D, Anderson localization occurs. The linear spectrum is purely discrete with exponentially localized eigenmodes. Then any initially localized wave-packet is a linear combination of countable localized eigenmodes. The time evolution of any local coordinate of the system is described by an absolutely convergent series of time periodic sine functions (which is called an almost periodic function). Such states are time recurrent, which means they return arbitrarily close to their initial condition. As a result, the amplitude of the wave-packet cannot go to zero. It does not spread and remains localized forever.

In disordered nonlinear systems where the linearized spectrum is purely discrete, each spatially localized time periodic solution persists but its frequency varies through the linear spectrum within a Cantor set with infinitely many gaps avoiding linear resonances (i.e. the Anderson eigenfrequencies). Actually, those solutions can be considered as DBs which are intraband and become standard (extra band) DBs when their frequency escapes the phonon spectrum. Numerical calculations of this was done in [Kopidakis & Aubry, 2000]. However, those solutions are not attractors as for the DBs in spatially periodic systems, where the linearized spectrum is absolutely continuous. The above argument for the spontaneous formation of a DB does not hold because even if the frequency of the DB belongs to the linearized spectrum, no linear radiation at infinity is possible. Otherwise, up to now no numerical evidence for the spontaneous formation of DBs in such disordered nonlinear systems with purely discrete linear spectrum has been observed. We recently suggested [Johansson *et al.*, 2010] that there are many more stationary solutions than DBs in such infinite systems which are almost periodic solutions or Kolmogorov–Arnol'd–Moser (KAM) tori. Those solutions do not exist in spatially periodic systems, so in that case, DBs are the only possible localized stationary solutions. The situation is quite different in disordered systems with a purely discrete linear spectrum.

Actually, up to now, the existence of these KAM tori was ignored. It was generally believed that in infinite systems (unlike finite systems), such regular stationary KAM solutions (if any) could not manifest in numerical tests because of a zero probability to occur. It was believed that the nonlinearities which couple the Anderson modes make most trajectories chaotic with a broad band spectrum. The expected consequence of this assumption is that the wave-packet should excite at any time nearby Anderson modes still at rest, and thus it should spread its energy diffusively through the whole system. While the wave-packet spreads, its amplitude decays so that the effective nonlinear coupling between the Anderson modes also decays. As a result, the second moment of its energy distribution is expected to diverge slower than in a standard diffusive process, so that a power law is expected as t^α with $\alpha < 1$ unlike standard diffusion where $\alpha = 1$. This diffusion was called "subdiffusion". Early numerical simulations were done [Shepelyansky, 1993; Molina, 1998; Pikovsky & Shepelyansky, 2008; Flach *et al.*, 2009; Skokos *et al.*, 2009] on some models and for some parameter ranges and apparently confirmed this conjecture over the longest possible evolution time available on computers. These tests gave essentially the same qualitative behavior for different models (mostly random DNLS models and random Klein–Gordon models).

However, controversy about a subdiffusive behavior was raised in several theoretical papers. Wang and Zhang [2009] conjectured on the basis of exact results at small nonlinearity on the long time behavior of the tails of a wave-packet, that if there is sub-diffusion, it could not be a power law of time but at most a logarithmic law of time. On the basis of an approach considering the nonlinearity as a perturbation, [Fishman *et al.*, 2009] could control the convergence of the perturbation series up to high orders with some assumptions which were checked numerically. They reached a similar conclusion at small nonlinearity.

This paper is partly a review of early results about the existence of KAM solutions in the random DNLS model [Johansson *et al.*, 2010] and partly comprises discussions and criticismes about these early results on the same model. We shall show first that in some domain of parameters and over comparable evolution time, there are many initially localized wave-packets which do not spread at all and behave like almost periodic (KAM) trajectories. We shall also confirm that the wave-packets which do not behave as almost periodic trajectories look initially chaotic but we shall prove that in some cases that complete spreading is impossible despite the trajectory being initially chaotic. This finding raises the question of the validity of the early explanation for spreading. Since at least in some proven cases spreading must stop, we suggest that the wave-packet though it looks to spread initially, may very slowly converge to a limit state with weak chaos and a singular continuous spectrum (instead of broad band i.e. absolutely continuous) in order to avoid all possible resonances with the nonexcited Anderson modes, thus stopping diffusion. This conjecture requires more mathematical and numerical investigations.

2. Early Results About KAM Tori in Infinite Disordered Systems with a Purely Discrete Linear Spectrum

Let us first discuss the possible existence of KAM trajectories in infinite nonlinear systems. The original KAM theory (for a simple presentation, see, e.g. [Gignoux & Silvestre-Brac, 2009]) predicts that, under generic conditions, Hamiltonian systems with a finite number of degrees of freedom, N, close enough to an integrable limit (e.g. the linear limit) exhibit *quasiperiodic* trajectories which are dense on invariant N-dimensional tori in a $2N$-dimensional phase space. The N fundamental frequencies of those trajectories depend on the torus. When some integer combination of the frequencies vanishes, resonance is obtained, and tori which are resonant or almost resonant generally break up into chaotic trajectories when the Hamiltonian is perturbed from the integrable limit. Since there are infinitely many possible resonances, infinitely many gaps appear densely in phase space. As the volume of the gaps drops exponentially with the order of the corresponding resonance, the persisting tori form a fat Cantor set (i.e. of nonvanishing Lebesgue measure), which goes to full measure at the integrable limit. There is no general extension of the KAM theory for infinite systems ($N = \infty$), except for some special models [Fröhlich *et al.*, 1986; Pöschel, 1990]. It was generally believed that most KAM tori disappear when the dimension of the dynamical system is infinite (see, e.g. [Froeschlé & Scheidecker, 1975]).

Simple empirical arguments confirm that KAM tori, which are spatially localized (square summable, l_2) cannot survive when the spectrum of the linearized system is *absolutely continuous* in some frequency interval. This situation occurs, e.g. for spatially periodic arrays of coupled anharmonic oscillators. In such models, any hypothetical quasiperiodic solution with at least two incommensurate frequencies would generate harmonics densely on the real axis, overlapping with the interval of the absolutely continuous part. Thus, these harmonics would radiate energy towards infinity, so that the localized energy could not be conserved [Aubry & Schilling, 2009] (by contrast, simple *periodic* solutions may remain localized, forming DBs which may be dynamical attractors for some initial conditions [Sievers & Takeno, 1988]). In some models quasiperiodic solutions with only two fundamental frequencies may also exist, providing their full spectrum is not dense and does not overlap the phonon band [Johansson & Aubry, 1997]. However, as we pointed above, these arguments do *not* hold when the linear spectrum is *purely discrete*. It is known, e.g. that spatially localized periodic solutions with frequencies *inside* the linear spectrum exist generically in systems with linear Anderson localization [Albanèse & Fröhlich, 1991; Kopidakis & Aubry, 2000].

A pioneering work [Fröhlich *et al.*, 1986] has proven the existence of KAM tori in infinite random nonlinear Hamiltonian systems. However because of the high complexity of KAM theories, for the sake of simplicity they have focused on simplified models. They consider random nonlinear models with a Hamiltonian which can be put in the form

$$\mathcal{H} = \sum_{i\in\mathbb{Z}^d} \frac{1}{2}(P_i + \omega_i^2 Q_i^2) + \epsilon \sum_{\langle i,j\rangle} f_{\langle i,j\rangle}(Q_i, Q_j, P_i, P_j) \tag{1}$$

where ϵ is a small parameter. In other words, they assume that the Anderson linear modes i can be arranged on a periodic d dimensional array $\mathbb{Z}^d$ with random frequencies ω_i obeying a smooth probability law $d\rho(\omega_i)$. They assume additionally that the nonlinear terms $f_{\langle i,j\rangle}(Q_i, Q_j, P_i, P_j)$ are of order 4 in P_i, Q_i and couples nearest neighbor $\langle i, j\rangle$ oscillators.

Their result can be stated simply. Consider an arbitrary solution of the uncoupled system ($\epsilon = 0$) which is highly localized or which decays faster than any exponential of the distance. It is proved that when the perturbation parameter ϵ is not too large, there is a nonvanishing probability over the choice of frequencies ω_i, such that there exists an exact almost periodic (KAM) solutions close to this unperturbed solution corresponding to an infinite dimension torus in the phase space. Moreover, this probability goes to 1 as the nonlinear perturbation ϵ goes to zero. In other words, for small nonlinear perturbation, an initially localized wave-packet has a finite probability to remain stationary and localized forever.

The FSW theorem does not state the measure of KAM tori is finite because in infinite systems, where the phase space has infinitely many dimensions, the concept of measure is undefined and cannot be used. They prove the existence of KAM tori only in terms of probability with respect to the disorder realization. However, we can define the concept of projected measure if we consider a family of initial conditions described by a finite number of parameters. Then those initial conditions which generate KAM tori could be measured in that finite dimensional subspace of parameters. This is what we do in Sec. 4 by considering the amplitude of the initial single-site excitation as the only parameter.

Another interesting rigorous result has been found [Bourgain & Wang, 2008] for the well-studied random DNLS model with the dynamical equation

$$i\dot{\psi}_n = (\epsilon_n + \chi|\psi_n|^2)\psi_n - C\Delta\psi_n, \qquad (2)$$

where n belongs to a lattice with arbitrary dimension d, the discrete Laplacian $\Delta\psi_n = \sum_{m:n}\psi_m$ is defined as the sum of ψ_m over the nearest neighbor sites m of n on $\mathbb{Z}^d$ and ϵ_n is a random variable with a smooth probability law.

Considering a trivial solution $\psi_n(t) = \psi_n(0)e^{-\epsilon_n t}$ at the uncoupled linear limit ($\chi = 0$ and $C = 0$) with a compact support that is with $\psi_n(0) \neq 0$ only for a finite number p of sites $n \in \mathbb{S} \subset \mathbb{Z}^d$, and $\psi_n(0) = 0$ for $n \notin \mathbb{S}$, they prove that when χ and C are not too large, this solution survives as a quasiperiodic solution (which correspond to a finite p-dimensional torus in the phase space) with a nonvanishing probability over the choice of ϵ_n.

Moreover, this probability goes to 1 when χ and C go to zero.

This theorem looks very similar to the FSW theorem, but it considers a specific well-studied model without simplifications. Otherwise, this theorem only proves the existence of invariant tori with finite dimension p (though it could be arbitrarily large) unlike the FSW where the invariant tori have infinitely many dimensions. The reason is that in finite systems, invariant tori with dimension $P < N$ generally have zero measure. We expect similarly the solutions found in [Bourgain & Wang, 2008] to have zero probability to occur in an infinite system. The consequence is that the probability to find an initial localized wave-packet generating such tori is zero.

Despite these theorems suffering such restrictions, they suggest that nondiffusive almost periodic wave-packets may exist with a finite probability. We conjecture that these theorems have extensions but these developments unfortunately involve complex and difficult KAM mathematics. However, we support our conjecture by empirical and numerical calculations on the 1D random DNLS model.

3. Resonances in the 1D Random DNLS Model

The evolution equation of model Eq. (2) in one-dimension is (see, e.g. [Shepelyansky, 1993; Molina, 1998; Pikovsky & Shepelyansky, 2008; Skokos *et al.*, 2009; Flach *et al.*, 2009]),

$$i\dot{\psi}_n = (\epsilon_n + \chi|\psi_n|^2)\psi_n - C(\psi_{n+1} + \psi_{n-1}). \qquad (3)$$

The total norm $\mathcal{N}$ of the solution defined by $\mathcal{N}^2 = \sum_n |\psi_n(t)|^2$ is time invariant. This equation has two integrable limits with trivial solutions at $\chi = 0$ (linear limit) and $C = 0$ (anticontinuous limit). The random onsite energies ϵ_n can be chosen as uniformly distributed in the interval $[-W/2, W/2]$. (See [Lahini *et al.*, 2008] for a direct experimental realization of Eq. (2) with optical waveguide arrays.) Then in this 1D model, the linear spectrum ($\chi = 0$) is discrete and the corresponding eigenstates are l_2 and exponentially localized.

We suggest that Eq. (3) may also sustain *infinite dimensional* invariant tori of *almost periodic* solutions with a nonvanishing probability.

Definition 3.1. An almost periodic function $f(t)$ in the Harald–Bohr sense (e.g. see [Bredikhina, 2002]), is defined as the sum of an absolutely convergent series of sine functions, $f(t) = \sum_n f_n e^{i\omega_n t}$, where the set of frequencies ω_n is *countable* and $\sum_n |f_n| < +\infty$.

Our empirical arguments can be summarized as follows: We consider an arbitrary l_2 small-amplitude initial condition, corresponding to a distribution of excited Anderson modes of the linear model. The perturbation by a weak nonlinearity affects very little the evolution of the linear systems during short times but may have drastic accumulation effects after long times. At the lowest order, this perturbation generates small oscillating norm currents between different Anderson modes. The norm variation of each Anderson mode obtained by time integration of the norm currents toward this mode is obtained as a series with possibly *small denominators*, which are integer combinations of four Anderson mode frequencies (since the nonlinearity is of order 4). In order that a perturbation calculation be consistent at all times, the amplitudes of the time integrated current oscillations should remain small in order that the variations of amplitudes of the Anderson modes also remain small at all time. We then have to check that the small denominators are always larger than their corresponding numerators (which can be also very small). When this condition is not fulfilled, we have resonance or almost resonance between four Anderson modes. Those modes would then slowly grow (or decay) as a function of time with large amplitude variations, escaping after a finite time from the regime of validity of the perturbative calculation.

When there is no resonance at all, the amplitude of the Anderson modes will remain almost constant at any time, and consequently perturbation theory is at least consistent at lowest order. To be rigorous, we should consider the higher order of the perturbation expansion, which would involve higher order resonances involving $6, 8, \ldots$ frequencies, but the calculation becomes too tedious. We assume that these resonances have much smaller probability of occurrence than the lowest order resonance, and neglect them. Note that if we could control perturbation theory at any order, we would finally attain a proof of KAM theory.

We estimate the probability that the nonlinear couplings almost induce resonances and we find *this probability vanishes in the limit of small norm* of the initial wave-packet. Thus at small norm, most wave-packets do not involve any resonance. If we assume that perturbation theory converges at all orders, most wave-packets should generate almost periodic solutions. Our argument explicitly uses the fact that the linear spectrum is discrete with exponentially localized eigenstates, and thus does not hold for a system with an absolutely continuous part in its spectrum. Moreover, we predict that the existence domain of KAM-like tori shrinks to zero in the limit of weak disorder, or equivalently when the shortest localization length diverges.

The spectrum of the linearized Eq. (2) is purely discrete (with probability 1) at any disorder since it is a 1D random onsite model. The associated linear operator possesses a (countable) set of eigenvalues ω_p associated with a basis of real l_2 eigenstates $\{\phi_n^{(p)}\}$ which are exponentially localized:

$$\omega_p \phi_n^{(p)} = \epsilon_n \phi_n^{(p)} - C(\phi_{n+1}^{(p)} + \phi_{n-1}^{(p)}) \tag{4}$$

$\{\psi_n(t)\}$ is expanded in this Anderson basis, $\psi_n(t) = \sum_p \mu_p(t)\phi_n^{(p)}$ which define the new time dependent complex variables $\mu_p(t)$. Then $|\psi_n(t)|^2 = \sum_{p,p'} \mu_p^\star(t)\mu_{p'}(t)\phi_n^{(p)}\phi_n^{(p')}$, and the norm square

$$\mathcal{N} = \sum_n |\psi_n|^2 = \sum_p |\mu_p|^2 \tag{5}$$

is the second conserved quantity of Eq. (2) (in addition to the Hamiltonian). Then, we obtain for the new complex coordinates μ_p

$$i\dot{\mu}_p = \omega_p \mu_p + \sum_{p'} C_{p,p'}(t)\mu_{p'}(t), \tag{6}$$

where $C_{p,p'}(t)$ is a real function of time through the coordinates themselves defined as

$$C_{p,p'}(t) = \chi \sum_{q,q'} \mu_q^\star(t)\mu_{q'}(t) V_{p,p',q,q'}. \tag{7}$$

Coefficients $V_{p,p',q,q'}$ are defined as the overlap sums $V_{p,p',q,q'} = \sum_n \phi_n^{(p)}\phi_n^{(p')}\phi_n^{(q)}\phi_n^{(q')}$. These coefficients are real and time constant but depend on the specific realization of the random potential. In the limit of small amplitudes of $|\mu_p|$, cubic terms in Eq. (6) are higher order and may be neglected at least during some time. The linear behavior is $\mu_p(t) \approx \mu_p(0)e^{-i\omega_p t}$. Then coefficients $C_{p,p'}(t)$ are almost periodic functions of time, since they are an absolutely convergent series of sine functions involving a countable set of periods,

$$C_{p,p'}(t) = \chi \sum_{q,q'} V_{p,p',q,q'}\mu_q^\star(0)\mu_{q'}(0)e^{i(\omega_q - \omega_{q'})t}. \tag{8}$$

We obtain from Eq. (6),

$$\frac{d}{dt}|\mu_p|^2 = 2\sum_{p' \neq p} C_{p,p'}(t)\mathrm{Im}(\mu_p^\star \mu_{p'}) \tag{9}$$

so that the norm current $J_{p' \to p}$ between two different Anderson modes $p \neq p'$ is

$$\begin{aligned} J_{p'\to p} &= 2C_{p,p'}(t)\mathrm{Im}(\mu_p^\star \mu_{p'}) \\ &= 2\chi \sum_{q,q'} [V_{p,p',q,q'}|\mu_q(0)| \\ &\quad \cdot |\mu_{q'}(0)|\ e^{i((\omega_q-\omega_{q'})t-(\alpha_q-\alpha_{q'}))}]|\mu_p(0)| \\ &\quad \cdot |\mu_{p'}(0)|\sin((\omega_p - \omega_{p'})t - (\alpha_p - \alpha_{p'})), \end{aligned} \tag{10}$$

where α_p is the initial phase of $\mu_p(0) = |\mu_p(0)|e^{-i\alpha_p}$. This current is almost periodic in time, and its oscillations should be small in order that $|\mu_p(t)|^2$ remains approximately constant, so that $\mu_p(t) \approx \mu_p(0)e^{-i\omega_p t}$ remains valid for all times. Since time integration of the current yields denominators $\omega_q - \omega_{q'} \pm (\omega_p - \omega_{p'})$ which may be small, this condition requires that there should be no strong resonances between any pairs of sites $p \neq p'$. We consider resonances to be "weak enough" when for all terms, the modulus of the numerator is smaller than the modulus of the denominator, i.e.

$$|\omega_q - \omega_{q'} \pm (\omega_p - \omega_{p'})| \gtrsim \kappa|\chi V_{p,p',q,q'}| \cdot |\mu_q(0)| \cdot |\mu_{q'}(0)| \quad (11)$$

for all modes q and q', where κ is parameter we can chose of order 1 (Chirikov criterion). Note also that $\pm$ can be dropped, since the same condition is obtained if q and q' are exchanged.

We assume now in order to fix the ideas that ω_p are independent random numbers distributed in some interval with a *smooth probability law* with maximum density P_0. Each resonance p, p', q, q' has thus a probability to occur $P_{p,p',q,q'}$ which is bounded as

$$P_{p,p',q,q'} < 2P_0\kappa|\chi V_{p,p',q,q'}| \cdot |\mu_q(0)| \cdot |\mu_{q'}(0)|. \quad (12)$$

The probability P_R (over the disorder realization) that *at least one resonance* exists in the system for this initial condition, is bounded by the sum of these probability bounds divided by 2 (resonance p, p', q, q' is equivalent to p', p, q', q). Then we can bound P_R as

$$P_R \leq P_0\kappa|\chi| \sum_{q,q'} |\mu_q(0)|A_{q,q'}|\mu_{q'}(0)| \quad (13)$$

where the coefficients $A_{q,q'}$ of the positive operator $\mathbf{A}$ are $A_{q,q'} = \sum_{p\neq p'} |V_{p,p',q,q'}|$. Then we obtain

$$P_R \leq P_0\kappa|\chi| \cdot \|\mathbf{A}\| \cdot \|\{\mu_q(0)\}\|^2,$$

with $\mathbf{A} = \{A_{q,q'}\}$ and the l_2 norm $\|\mathbf{A}\| = \sup_{\mathbf{X}}(\|\mathbf{A}\cdot\mathbf{X}\|/\|\mathbf{X}\|)$ may be equivalently defined as the smallest upper bound of the eigenspectrum of $\mathbf{A}$.

This upper bound for the probability to have at least one resonance is thus directly related to the l_2 *norm* square $\mathcal{N}^2 = \sum_n|\psi_n|^2 = \sum_p|\mu_p|^2$ of the initial condition (this was also found numerically in [Skokos & Flach, 2010]). The probability $P_N = 1 - P_R$ to have *no resonance* thus has a nonvanishing *lower* bound,

$$P_N \geq 1 - P_0\kappa|\chi| \cdot \|\mathbf{A}\| \cdot \mathcal{N}^2 \quad (14)$$

when the norm $\mathcal{N}$ of the initial condition is small enough ($\|\{\mu_q(0)\}\|^2 < 1/(P_0\kappa|\chi| \cdot \|\mathbf{A}\|)$), we obtain $P_N > 0$.

Consequently, if the norm $\|\mathbf{A}\|$ is *not infinite*, the probability to have no resonance will be *nonvanishing*, and go to 1 when the norm of the initial condition goes to zero. It remains to prove that the norm of $\mathbf{A}$ is finite if and only if the linear spectrum is purely discrete. An upper bound for $\|\mathbf{A}\|$ can be obtained from

$$\begin{aligned}\|\mathbf{A}\| &\leq \sup_q \sum_{q'} |A_{q,q'}| \\ &\leq \sup_q \sum_{p\neq p',q'} \sum_n |\phi_n^{(p)}| \cdot |\phi_n^{(p')}| \cdot |\phi_n^{(q)}| \cdot |\phi_n^{(q')}| \\ &< \sup_q \sum_n \left(\sum_p |\phi_n^{(p)}|\right)^3 |\phi_n^{(q)}|. \end{aligned} \quad (15)$$

If the eigenstates are exponentially localized, then $\sum_p |\phi_n^{(p)}| < +\infty$ and $\sum_n |\phi_n^{(q)}| < +\infty$. If we assume these sums are bounded for all n or q by the same constant S, then $\|\mathbf{A}\| < S^4 < +\infty$. If $\mu_q(0)$ is not arbitrarily chosen, we may get a better upper bound for the existence of KAM tori. For example, if $\mu_q(0) = \delta_{q,q_0}\|\{\mu_q(0)\}\|$ is initially localized at a single Anderson mode $q' = q_0$ [Flach *et al.*, 2009], we have $P_R \leq P_0\kappa|\chi|(\sup_{q'} |A_{q',q'}|) \cdot \|\{\mu_q(0)\}\|^2$ where $\sup_{q'} |A_{q',q'}| < \|\mathbf{A}\|$. Thus, initial wave-packets which are close to small amplitudes of single Anderson modes survive much better as almost periodic solutions than those which are arbitrarily spread (in Anderson space) at the same norm. This effect is especially important when the localization length is large since then $\sup_{q'} |A_{q',q'}| \lll \|\mathbf{A}\|$. The same is true if the initial wave-packet is split into several packets with smaller norm which are far apart at the scale of the localization length.

Note also that the bound Eq. (14) may be improved for any initial norm if we assume that the wave-packet is initially well spread, that is $|\psi_n(0)|^2$ is small for any n. Then, the probability that resonance p, p', q, q' does not occur is $1 - P_{p,p',q,q'}$ where $P_{p,p',q,q'}$ in Eq. (12) is small. Then the probability that no resonance at all occurs is $P_N \geq \prod_{p,p',q,q'}(1 - P_{p,p',q,q'}) \approx e^{-\sum_{p,p',q,q'} P_{p,p',q,q'}}$ which yields the improved bound compared to Eq. (14)

$$P_N \geq e^{-P_0\kappa|\chi|\cdot\|\mathbf{A}\|\cdot\mathcal{N}^2} \quad (16)$$

which is valid in the limit $\sup_n |\psi_n(0)|^2$ small.

Note that since $\sum_n|\phi_n^{(q)}|^2 = 1$, if the localization length increases and diverges, then $S > \sum_n|\phi_n^{(q)}| \to \infty$. To obtain a reasonable estimate of $\|\mathbf{A}\|$ we assume an exponential bound for all eigenstates,

$$|\phi_n^{(p)}| < K\sqrt{\frac{1-\lambda^2}{1+\lambda^2}}\lambda^{|n-p|},$$

where K is some constant and $\lambda = e^{-1/\xi}$, where ξ is the localization length (though for random systems, this bound cannot be uniform, choosing K large should make valid such a bound for most eigenstates). Then $S^4 = K^4((1+\lambda)^6/(1+\lambda^2)^2)(1/(1-\lambda)^2)$.

When the localization length diverges at weak disorder we find

$$\|\mathbf{A}\| \lesssim 16K^4\xi^2.$$

Consequently this upper bound for the norm diverges, suggesting that $\|\mathbf{A}\|$ might also diverge in the same way. Moreover, the norm $\|\mathbf{A}\|$ can be explicitly calculated for finite systems with periodic boundary conditions and size N where the eigenstates are normalized plane waves. We readily find $\|\mathbf{A}\| = N-1$; which confirms that $\|\mathbf{A}\|$ diverges in the limit of large size N.

We did not consider the probabilities of resonances at higher orders $6, 8, \ldots$ which are cumbersome to calculate. The correction on the bound of P_R would be higher order in $\|\{\mu_q(0)\}\|^2$ and thus could be neglected in the limit of small norm. We conjecture that at each order $2p$, these probabilities can also be bounded by convergent series multiplied with $\|\{\mu_q(0)\}\|^{2(p-1)}$. The probability of higher order resonance is expected to decay exponentially with the order. Thus when the norm of the initial condition is not too large, we would expect that the probability of *no resonance at any order* is *nonvanishing* and still bounded from below, going to 1 as the norm goes to zero. This conjecture seems compatible with the conjectures of [Wang & Zhang, 2009] and [Fishman *et al.*, 2009] which were suggested either by exact results or by numerical tests.

Note that for finite systems with size N, the linear spectrum is always discrete and the series for bounds P_R become finite sums, implying $\|\mathbf{A}\| < +\infty$. Then, we know that the conclusion of our empirical argument is consistent with KAM theory, predicting the existence of N-dimensional invariant tori of quasiperiodic solutions at small enough amplitude (or equivalently, for finite systems, small enough norm) with a probability going to 1 at zero amplitude.

Our conjecture is that this argument also holds for infinite systems provided $\|\mathbf{A}\| < +\infty$. This situation occurs when the linear spectrum is purely *discrete* with exponentially localized eigenstates, but is not fulfilled when it contains an absolutely continuous part. Then we would conclude, that the norm region for initial conditions where KAM tori may exist shrinks to zero when the localization length diverges, approaching the limit without disorder where the linear spectrum is absolutely continuous and no l_2 almost periodic exact solution could survive due to radiation.

Krimer and Flach [2010] recently investigated the statistics of resonances on many disorder realizations. They found that resonances are mostly found within the localization volume where the elements $V_{p,p',q,q'}$ are non-negligible. Moreover, they found the probability that an Anderson mode excited to a norm square n is resonant (within some group of four modes in the localization volume excited with similar norms), is $\mathcal{P}(n) \sim 1 - e^{-\chi nc}$ where c only depends on the amplitude W of the disorder.[1] They found that $c(W)$ becomes small for large disorder parameter W and large for small W. The probability of nonresonance in a wave-packet exciting several Anderson modes is simply another exponential $\sim e^{-\chi N^2 c(W)/4}$ of the total norm square $\mathcal{N}^2$ (since resonances are counted four times). These quantitative results nicely agree with our result Eq. (16) that there is a nonvanishing probability to find nonresonant wave-packets for nonvanishing W. The domain of norm where this probability is non-negligible shrinks to zero when the disorder W becomes small (long localization length), and diverges when W is large (short localization length).

4. Method for the Numerical Investigations of KAM Tori

We now test these empirical arguments for the existence of KAM tori. Attempting to distinguish numerically KAM tori among other solutions in the random DNLS equation, we first note that the components of trajectories generating KAM tori are almost periodic functions of time. In practice, it is not possible to calculate for checking the time Fourier transform of any component over very long time evolution if it is a sum of Dirac functions.

4.1. *Bohr recurrences*

Test over short time would yield a very poor accuracy for the resolution of these Dirac peaks. It is

[1]Krimers and Flach call norm what is actually the square of the norm. This does not change their conclusion when this minor correction is done.

much simpler and more elegant to test almost periodicity by using the Harald Bohr theorem:

Theorem 1. *If a function $f(t)$ is almost periodic (according to Definition 3.1), then for any $\varepsilon > 0$, there exists a relatively dense set of translations τ_n such that $|f(t) - f(t+\tau_n)| < \varepsilon$ for all $t \in]-\infty, +\infty[$. Conversely, if a function fulfills this property, it is an almost periodic function.*

(Relatively dense means that for the ordered sequence $\tau_n < \tau_{n+1}$, the difference $0 < \tau_{n+1} - \tau_n < B$ is bounded from above for all n by a constant B.) Thus, for KAM-like tori, *recurrences* should be observed numerically for all quantities (which are almost periodic functions of time) such as local coordinates at arbitrary sites, momenta, participation number, etc. Our observations show that if recurrences appear for one of these quantities, we also find them for any other.

For having Bohr recurrence, we have to check that the initial trajectory and the time translated trajectory by a pseudo period remains close to each other at all time within an accuracy of the order of ε. Thus Bohr recurrence is much more compelling than Poincaré recurrence, which only requires that the trajectory returns close to its initial state without requiring the time-translated trajectory and the initial one remain close to one other at all times. Though the Poincaré theorem states that in any Hamiltonian system with a finite number of degrees of freedom, most trajectories (with probability 1) are recurrent, this theorem does not hold anymore when the number of degrees of freedom is infinite. The reason is that initial wave-packets in an infinite system which are not Bohr recurrent (and thus do not generate KAM tori), spatially spread and thus never return close to their initial condition. In our systems which are not infinite but only very large, we never numerically observed simple Poincaré recurrences without experiencing a Bohr recurrence.

Some numerical problems with this method are as follows:

(i) the Harald–Bohr theorem can be checked only for finite times $\tau \in [0, T]$ and $t \in [0, T]$, where T is the time of integration and for finite-size systems,

(ii) ε cannot be chosen too small to avoid recurrences from becoming too rare or disappearing over the integration time since the corresponding pseudo period diverges as ε goes to zero. Actually, when decreasing ε, we still observe pseudo periods but they do become sparser and sparser,

(iii) the system size should be sufficient, in order that the amplitude is practically zero at the edge, so that boundary effects can be neglected during the integration time T,

(iv) the integration accuracy should be good enough for avoiding numerical drift from KAM-like tori to neighboring KAM-like tori or nearby chaotic trajectories belonging to the gaps. In practice, the relative error in the conserved quantities are kept at the order 10^{-6} or smaller in all simulations, with consistency checks for smaller systems reaching accuracies 10^{-8}–10^{-10}.

If there are KAM tori persisting over infinite time, we should expect that their structure as some model parameter varies is a *fat Cantor set with infinitely many gaps* due to resonances at all orders, but keeping a *nonvanishing Lebesgue measure*. Resonances involving four far distant Anderson modes as well as higher order resonances involving more Anderson modes, generate smaller and smaller gaps so that the total width of these gaps remains finite and smaller than the full measure. The consequence is that the complementary Cantor set also has nonvanishing measure.

Thus in numerical tests, the probability (in the space of initial conditions) that Bohr recurrence is found over a given time T, does *not* shrink to zero as T becomes very large. However, there are also *sticking trajectories* which were intensively studied by Contopoulos *et al.* (e.g. see [Contopoulos *et al.*, 1997; Contopoulos & Harsoula, 2010]). Those trajectories have initial conditions very close to KAM tori or to small gap Cantori (Cantori are unstable invariant subsets left after the destruction of KAM tori, they have small gaps nearby the critical point where the corresponding KAM torus is destroyed). These sticking trajectories behave like KAM tori over very long time (and in particular, have the Bohr recurrence property) before escaping and becoming clearly chaotic. Actually, sticking trajectories are upper critical manifestations of KAM tori which have been just (*weakly*) destroyed.

This sticking time for escape may exceed our computing time if they are very close to KAM tori. Because of this effect, small gaps in the Cantor set of KAM tori may be nonobservable over finite time computing but we also expect in that case their width is negligible. The practical consequence is that during our finite computing time, we should be able to distinguish only a finite number of gaps corresponding to the strongest resonances, which nevertheless yields a reasonable approximation of the Cantor set.

We indeed find some trajectories which exhibit Bohr recurrence over relatively long times, before they blow up as chaotic trajectories. Such trajectories were numerically identified in [Skokos *et al.*, 2009] as belonging to a "regime I" of rather small norm and/or strong disorder. Generally, all trajectories which remain recurrent after a time T will be termed *T-recurrent*.

An illustration is given in Fig. 2, showing the last observed recurrence time for a wave-packet initially localized at a single site ($\psi_n(0) = 0$ for $n \neq n_0$) in a particular disorder realization $\{\epsilon_n\}$ at rather strong disorder $W = 20$. The recurrence over the time evolution is observed on the norm at the initial site $|\psi_{n_0}(t)|^2$ and tested for a series of initial norm $|\psi_{n_0}(0)|^2$ we vary by small steps from 0 to 20. We choose the initial-site energy ϵ_{n_0} rather close to the upper band edge, in order that an increasing negative nonlinearity will "scan" the most possible important resonances inside the band. Any horizontal intersection of this graph at a given time T yields the set of initial norm generating T-recurrent trajectories. As can be seen, many trajectories remain recurrent for times larger than 10^7. We can also see that inside the main gap there are also many sticking trajectories over times which range from 10 up to 10^6. Those trajectories are likely very close to small gap Cantori. We also note that the number of trajectories which are still sticking over times which are shorter than the computing time (and thus are not KAM tori) decays as this time becomes longer. We suggest to check the sticking trajectories by increasing slightly the disorder parameter W (for the same disorder realization). Then we should observe the increase of the sticking time for those sticking trajectories and for some of them this time could go beyond the computing time, suggesting that some KAM tori are restored.

4.2. *Domain of KAM tori*

Figure 3 shows the existence domain of Bohr recurrent trajectories as a function of the disorder parameter W and the initial norm $\mathcal{N}$ of the single site wave-packet. This domain presumably corresponds with a good approximation to the domain of existence of KAM tori. In addition to the regime of small norm (here $\mathcal{N} \lesssim 3.6$) expected from our empirical argument above, there is also a regime of recurrent states for $\mathcal{N} \gtrsim 19$, as well as a small interval around $\mathcal{N} \approx 15.6$. The recurrent trajectories in the large-norm regime may be interpreted by the fact that above some threshold norm, the

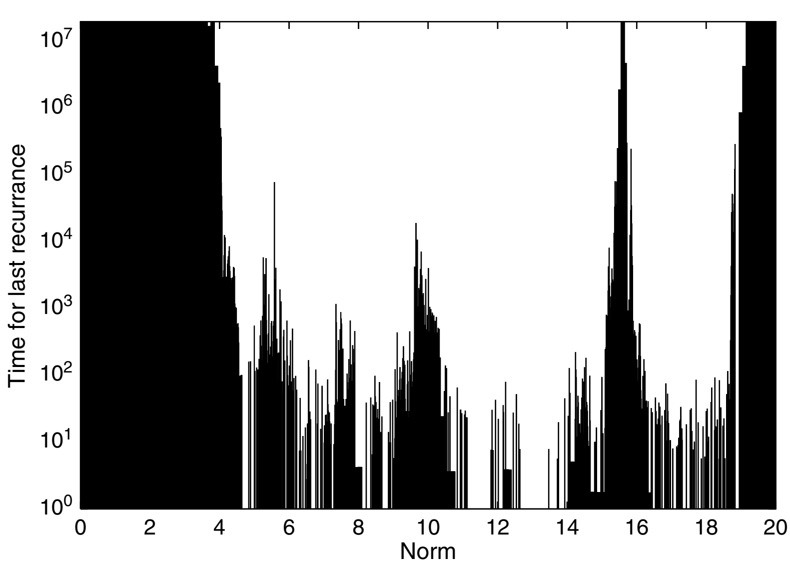


Fig. 2. Last observed time for recurrence ($\varepsilon/\mathcal{N} = 0.02$) in $|\psi_{n_0}|^2$ versus norm for the initial single site wave-packet $\psi_n(0) = \sqrt{\mathcal{N}}\delta_{n,n_0}$ in a particular disorder realization with $W = 20$. In this figure and Figs. 3–5, the disorder realization is the same and we use the same initial condition $\psi_n(t = 0) = \sqrt{\mathcal{N}}\delta_{n,n_0}$ with $\epsilon_{n_0} \approx 0.46529W$, $C = 1, \chi = -1$, and system size $N = 500$. Only W is varied.

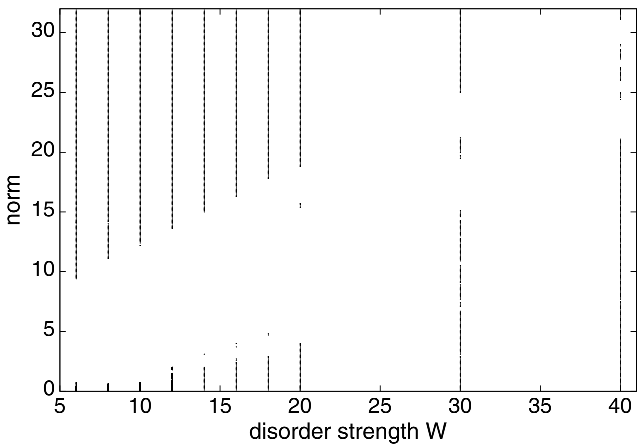


Fig. 3. Sets of T-recurrent trajectories at $T = 10^4$ versus the norm $\mathcal{N}$ for various disorder strengths W. The numerical step size $\varepsilon/\mathcal{N}$ for the norm varies from 0.1 ($W = 6$) to 0.005 ($W \geq 14$). Same disorder realization, initial condition and parameters as in Fig. 2.

main part of the norm will self-trap as a linearly stable extra band DB centered at the initial site n_0 [Kopidakis *et al.*, 2008], while the rest of the norm appears as a small perturbation to this DB. Small perturbations to this DB may be treated linearly which defines a linear spectrum. This spectrum is purely discrete with a countable basis of localized eigenmodes as well as for the linear random system without DB. This perturbation can be expanded on this basis of weakly excited DB eigenmodes. Then with a similar argument as above, we could expect that under the effect of weak nonlinearities which were neglected, these excited modes could persist as almost periodic (KAM) solutions, providing the absence of resonances (which have a small probability to occur).

Note also that the simultaneous limit of strong disorder and large norm, $W \to \infty$, $\mathcal{N} \to \infty$ with $\chi\mathcal{N}/W$ finite, is equivalent (rescaling time) to the anticontinuous (uncoupled) limit $C = 0$ which is also integrable, and where the linear localization length is vanishing. At this limit, any wave-packet motion decomposes into independent motions where each anharmonic oscillator n oscillates periodically with a frequency $\omega'_n \neq \epsilon_n$. These frequencies though they are different from the corresponding onsite linear frequency, are generally nonresonant with each other. Similarly to the hypothesis of the FSW theorem, if an only finite number of those oscillators are excited while the excitation of the far oscillators decays fast enough, we may thus expect a KAM-like regime up to a coupling C not too large (as for finite systems). The measure of those KAM tori should go to full measure when C goes to zero.

Thus, although numerics cannot provide a rigorous proof, the most plausible interpretation is that there is an underlying Cantor set of initial conditions generating KAM tori, persisting over infinite time and infinitely large systems. Note also that the gap structure of Fig. 2 is reminiscent of the "stickiness" phenomenon in low-dimensional systems, where many initial conditions close to (but outside) the Cantor set of KAM tori remain close for long times before they finally escape (compare, e.g. with Figs. 7 and 10 in [Contopoulos, 1997] for the standard map).

Figure 3 shows the variation of the sets of T-recurrent trajectories versus the disorder strength W over a rather modest time $T = 10^4$. As predicted, there is always a small-norm regime where most trajectories are T-recurrent, the size of which grows with increasing disorder strength. There is also always a high-norm T-recurrent regime with lower boundary increasing with increasing disorder, since the norm necessary for efficient self-trapping increases linearly with W for a single-site initial condition [Kopidakis *et al.*, 2008]. In-between these two regimes, for larger W there are also several intermediate regimes of T-recurrence, separated by gaps with nonrecurrent (chaotic) trajectories. For smaller W the relative sizes of these gaps grow, and they merge into one single main gap of trajectories which typically show a chaotic time-evolution and spread subdiffusively [Shepelyansky, 1993; Molina, 1998; Pikovsky & Shepelyansky, 2009; Skokos *et al.*, 2009] (for long but possibly finite times).

Figures similar to Fig. 3 can be obtained for larger times, although obtaining a good resolution with sufficient numerical accuracy to clearly identify recurrences makes it very time-consuming for times larger than $\sim 10^6$. As a rule of thumb, to determine persistent recurrences ε is divided by two for each order of magnitude in time. In Fig. 4, we give an example showing how the measure of the set of T-recurrent initial conditions varies with T, for different strengths W of the same disorder realization. In order to obtain a finite set, we here exclude recurrent trajectories belonging to the high-norm (self-trapped) regime, and moreover for comparison, we normalize the sets by dividing by the number of T-recurrent trajectories at $T = 10^3$. The data of Fig. 4 suggest the existence of an asymptotic set with a nonvanishing measure at infinite

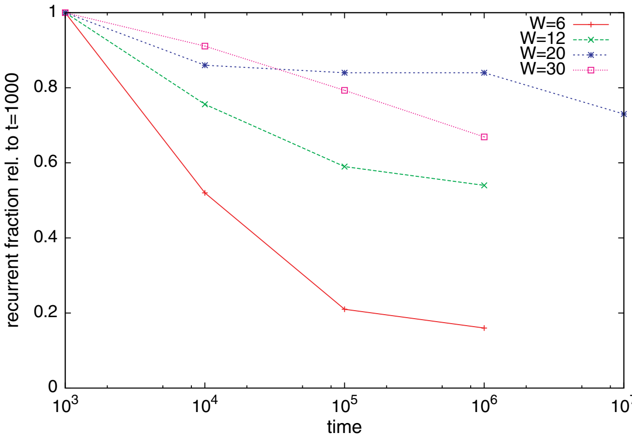


Fig. 4. Fraction versus time T of the trajectories T-recurrent at $T = 10^3$ (excluding the high-norm self-trapped regime) which remain T-recurrent also at longer times, for various disorder strengths W. Same disorder realization, initial condition and parameters as in Fig. 2. Note its variation becomes more substantial as W decays (weaker disorder).

time. However, for stronger disorder, it is clear from Fig. 4 that to get a clear picture of the asymptotic measure of this set, considerably longer integration times would be needed. This is due to the fact that the fraction of sticking trajectories, which are T-recurrent over very large but finite times, increases with the disorder strength (cf. Fig. 2). This fact is related to the appearance of many visible narrow gaps as the system becomes close to the integrable limit $C = 0$.

4.3. *Lyapunov coefficients*

Thus, the numerical study of T-recurrent trajectories for finite times can give us only an indication about the true nature of the KAM-like trajectories, and in particular the presence of long-time sticking trajectories makes it extremely difficult, e.g. to resolve the Cantor-set structure of resonances in the low-norm regime within a reasonable amount of computer time. We therefore now turn to discuss another technique to numerically distinguish KAM tori, in terms of the *tangent map* and the corresponding finite-time Lyapunov exponents. Small perturbations of Eq. (2) yield the Hill equation

$$i\dot{\eta}_n = (\epsilon_n + 2\chi|\psi_n|^2)\eta_n + \chi\psi_n^2\eta_n^\star - C(\eta_{n+1} + \eta_{n-1}). \quad (17)$$

If Eq. (2) possesses almost periodic solutions with the discrete set of frequencies $\omega_1, \omega_2, \ldots, \omega_p, \ldots$, corresponding to regular KAM tori (with full dimension), it can be written as

$$\psi_n(t) = F_n(\omega_1 t + \alpha_1, \omega_2 t + \alpha_2, \ldots, \omega_p t + \alpha_p, \ldots; \omega_1, \omega_2, \ldots, \omega_p, \ldots) \quad (18)$$

where $F_n(x_1, x_2, \ldots, x_p, \ldots; \omega_1, \omega_2, \ldots, \omega_p, \ldots)$ is analytic and 2π-periodic with respect to $x_1, x_2, \ldots, x_p, \ldots$, then $\eta_n(t) = \partial F_n/\partial x_p$ is an almost periodic solution of Eq. (17). Despite the set of solutions $\{F_n(x_1, x_2, \ldots, x_p, \ldots;\ \omega_1, \omega_2, \ldots, \omega_p, \ldots)\}$ is only defined for a fat Cantor set of frequencies $\{\omega_i\}$ with gaps around all resonances between finitely many frequencies, these functions are differentiable for most $\{\omega_i\}$ in this fat Cantor set except for those which are at gap edges. Those marginal tori are described in Eq. (18) by nonanalytic functions of the phases $\{\alpha_i\}$. These marginal KAM tori have zero measure in finite size systems.

Then, for most KAM tori with frequencies $\{\omega_i\}$, one gets a complete basis of solutions of Eq. (17) by adding also the partial derivatives with respect to frequencies ω_p

$$\eta_n(t) = \frac{dF_n}{d\omega_p} = t\frac{\partial F_n}{\partial x_p} + \frac{\partial F_n}{\partial \omega_p} \quad (19)$$

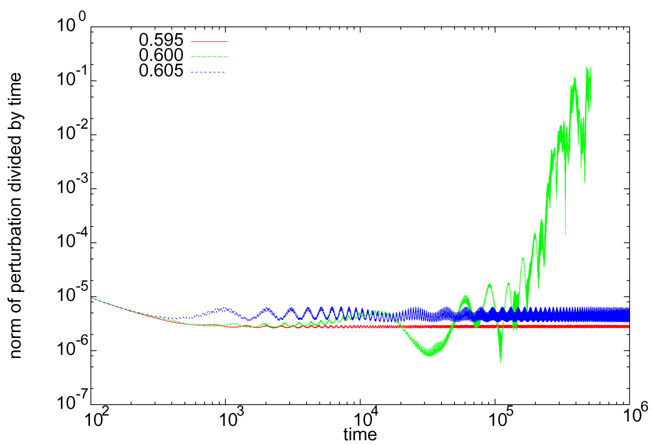
where also $\partial F_n/\partial \omega_p$ is also almost periodic.

Let us recall that regular KAM tori have dimension N in a system with $N + N$ degrees of freedom. When N is infinite, we can choose as a definition that KAM tori are regular when this base is complete. Invariant tori with finite dimensions may exist but are not regular KAM tori. For example, when N is infinite, the invariant tori predicted by [Bourgain & Wang, 2008] have only a finite number of dimension and do not generate a complete base by phase and frequency differentiation of Eq. (18).

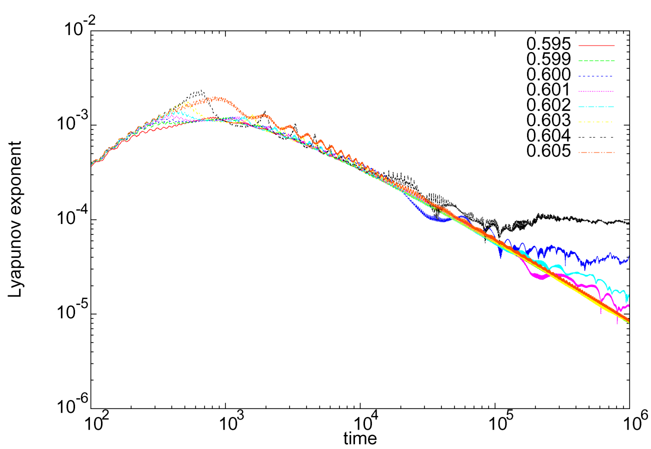
Consequently, if the solution of Eq. (2) corresponds to a regular KAM torus, the general solution of Eq. (17) which is a linear combination of solutions of the basis, should grow *linearly* as a function of time but not exponentially. Thus, KAM tori are linearly stable solutions unlike chaotic trajectories. (Linear stability is not granted for nonregular tori. Since the base is not complete. there are other directions in the phase space where instability could manifest (*whiskered tori*), which yields an exponential growth for most solutions of the Hill equation Eq. (17).)

For any solution ψ_n to Eq. (2), there is a trivial solution $\eta_n = i\psi_n$ to Eq. (17) corresponding to a global phase rotation. Numerically, we remove this component by subtracting the projection of

$\eta_n(t)$, obtained by integrating a randomized initial condition $\eta_n(0)$, on this vector. Then, considering the total norm of the perturbation divided by t, $(1/t)\|\eta(t)\|$ where $\|\eta(t)\| = \sqrt{\sum_n |\eta_n(t)|^2}$, this quantity exhibits bounded oscillations for all times if $\psi_n(t)$ corresponds to a KAM torus, and otherwise diverges exponentially with a positive Lyapunov exponent for any chaotic trajectory. A numerical illustration is given in Fig. 5(a), showing a narrow resonance gap in the low-amplitude KAM-like regime around $\mathcal{N} = 0.600$ for disorder strength $W = 12$.



(a)



(b)

Fig. 5. (a) Total norm of the Hill equation solution $\eta_n(t)$ divided by time, for three solutions corresponding to single-site initial conditions ψ_n with slightly different norms $\mathcal{N} \approx 0.6$. (b) Finite-time Lyapunov exponents for eight solutions in the same regime as in (a). At time 10^6, the upper curves in (b) correspond, from top to bottom, to $\mathcal{N} = 0.604, 0.600, 0.602, 0.601$, while the lower curves for $\mathcal{N} = 0.595, 0.599, 0.603, 0.605$ all follow very closely a curve $\sim \log t/t$, as expected for KAM tori. Disorder strength $W = 12$, other parameters and disorder realization are same as in Fig. 2.

From the so-obtained numerical solutions $\eta_n(t)$, we calculate finite-time Lyapunov exponents as $\Lambda(t_M) = (1/t_M)\sum_{m=1}^{M} \log(\|\eta(t_m)\|/\|\eta(t_{m-1})\|)$, where $t_0 = 0$, and t_m are chosen to correspond to a Poincaré section defined by $|\psi_{n_0}(t)|^2$ having a local maximum at each $t = t_m$. Thus, for a recurrent trajectory, the optimal recurrence times τ_k form a subset of t_m. Moreover, for an almost-periodic trajectory with a time-linear growth of $\|\eta\|$, $\Lambda(t)$ should decrease asymptotically to zero as $\Lambda(t) \sim \log(t)/t$ for large times. As can be seen from the example in Fig. 5(b), the behavior of $\Lambda(t)$ in the neighborhood of sharp resonances is very sensitive to small parameter variations. In this example, the trajectories are sticking trajectories which become possibly weakly chaotic after long time evolution in the interval $0.600 \leq \mathcal{N} \leq 0.602$ and in an even narrower interval around $\mathcal{N} \approx 0.604$, while apparently almost-periodic trajectories (with no visible deviations from the asymptotic behavior $\Lambda(t) \sim \log(t)/t$ for times larger than 10^6) are seen e.g. for $\mathcal{N} = 0.595, 0.599, 0.603$ and 0.605. Thus, this again supports the existence of a finite-measure Cantor set of almost-periodic KAM tori.

Comparing the numerical results from the analysis of recurrences and tangent map, they are consistent in the sense that when recurrences are lost, there is a clear deviation in $\|\eta\|$ from time-linear growth, and in all observed cases the growth is exponential with well-defined nonzero $\Lambda(t)$ (at least for long times). However, generally (and in particular, for strong disorder with many sticking trajectories), the tangent-map criterion is considerably more sensitive, and may signal a chaotic behavior several orders of magnitude in time before recurrence is finally lost.

4.4. *KAM tori in finite size systems*

All Bohr recurrent (KAM) trajectories were found to be exponentially localized, decaying essentially with the largest linear localization length ξ (e.g. $\xi_{\max} \approx 1.1$ for $W = 12$ as in Fig. 5) in a similar way as in the purely linear case, despite those trajectories cannot be described as linear combination of Anderson modes.

Thus we may expect that the KAM-like trajectories observed in large systems should be only slight perturbations of true KAM tori existing for finite chains when their size exceeds a few longest localization lengths. Indeed, it is well known that

KAM theory holds for finite size systems which do sustain KAM tori with finite probability.

If in those systems, there are KAM tori which are well-localized and almost vanishing at the edges of the sample, imbedding this solution in an infinite size system may be considered as a small perturbation. It is indeed proven in [Bourgain & Wang, 2008] that most finite size KAM tori persist for weak nonlinearity and weak coupling.

Though in the infinite system there are infinitely many new but almost unexcited Anderson modes outside the core of the solution, their nonlinear coupling with the core is small and drops very fast versus their distance. In principle infinitely many new possible resonances become possible but due to small coupling, they only generate small gaps in the frequency domain of the finite size KAM tori. The consequence should be that the major part of those tori survives to the system extension at infinity. Beyond a certain size, no difference with the infinite system is numerically observable.

For checking this conjecture, we study the measure of KAM tori generated by single site initial wave-packets in finite size systems as their size grows. Actually, since these KAM tori are well-localized, their domain of existence sharply depends on the local realization of the disorder that is mostly on ϵ_{n_0} at the single site initial excitation and also the few values of ϵ_n involved in the occupied region of the chain. In some realizations, these values may be well grouped and nearly resonant (local Anderson modes would be more extended) or in some others more sparse and far from resonance with peaked local Anderson modes. It is indeed observed that different realizations of the disorder in a small volume near the excited site can generate huge fluctuations in the measure of KAM tori. It is thus convenient to compile statistics on disorder realizations. This numerical investigation was not possible for the large size systems studied above, but fortunately it becomes feasible for small size systems. Moreover, we now know that for finding the KAM tori generated by single site wave-packets, it is useless to increase the system size beyond a few localization lengths.

An illustration is given Fig. 6 where, for $W = 12$ and single-site initial conditions $\psi_n(0) = \sqrt{\mathcal{N}}\delta_{n,n_0}$ with $0 < \mathcal{N} \leq 20$, we calculate the fraction of the total number of trajectories remaining T-recurrent after $T = 2 \cdot 10^5$, for chain lengths with $3 \leq N \leq 40$. For each chain length, we used 100 different (independent) disorder realizations, and determined ensemble averages and standard deviations (note that for small chains, ensemble statistics can be compiled with reasonable computational effort).

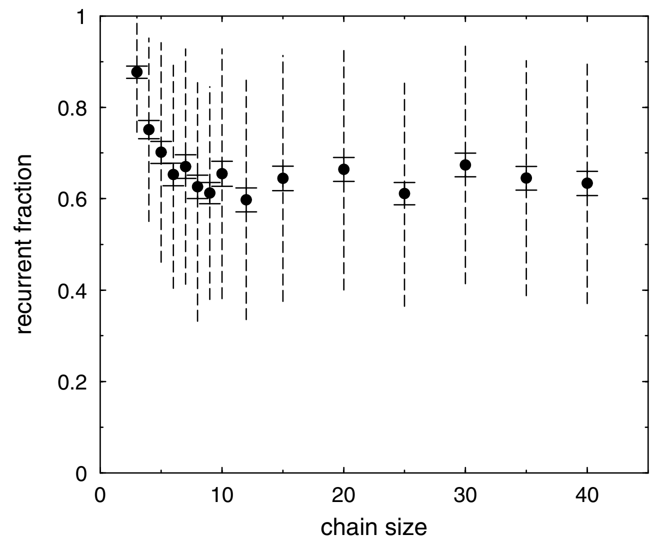


Fig. 6. Ensemble-averaged fractions of trajectories T-recurrent at $T = 2 \cdot 10^5$ versus chain size, for random single-site initial conditions with $0 < \mathcal{N} \leq 20$ and disorder strength $W = 12$. 100 different disorder realizations were used for each size, and dashed lines show the standard deviations of the distributions. Trajectories were considered T-recurrent if $\Lambda(T) < 10^{-4}$ (cf. Fig. 5).

As can be seen, this fraction already converged to its large-chain limit for $N = 6$ ($\approx 5\xi_{\max}$). We confirm also the expected large spreading of the results between different realizations. If ϵ_{n_0} is randomly chosen close to the linear band bottom, self-trapping occurs already for small values of $\mathcal{N}$, and therefore many trajectories will be KAM-like while in the opposite case, many resonances are encountered.

Indeed, for the particular realization used for the large-size simulations in Figs. 2–5, we obtain for $W = 12$ a total fraction of 38% T-recurrent trajectories in the interval $0 < \mathcal{N} \leq 20$ after $T = 10^5$ (decreasing to 37% at $T = 10^6$), essentially coinciding with the lower bounds in Fig. 6 (i.e. this realization does not behave exceptionally).

5. Discussion About the Long Time Behavior Non-KAM Wave-Packets

In summary, our empirical and numerical arguments suggest that infinite random DNLS models of Eq. (2) with purely discrete linear spectrum and finite energy solutions, exhibit two kinds of initial wave-packets both occurring with nonvanishing probability: (i) those generating almost periodic solutions (KAM tori) which are spatially localized solutions with exponential localization; and (ii) wave-packets which are apparently chaotic at least during long time.

In finite size systems (with N degrees of freedom), the Arnol'd conjecture [Arnol'd, 1964] though not yet proven, is well admitted. The dimension of the phase space is $2N$ but considering the invariants, the trajectories remain on an invariant submanifold with smaller dimension. In DNLS models, there are two invariants (the energy and the norm) and the dimension of this manifold is $2N - 2$. The dimension of the KAM tori is N. When N is sufficiently large, each KAM tori does not split the invariant manifold into two invariant subspaces. This is achieved in our random DNLS model when $2N - 2 > N + 1$ or $N > 3$. Then, the Arnol'd conjecture is that there is a unique chaotic region which is the complementary of the union of all KAM tori. Most chaotic trajectories (with probability 1) become dense with the same density measure in that region after an infinite time. This chaotic region contains all the resonance gaps between the KAM tori (Arnol'd webs) so that for any KAM torus, there are infinitely small perturbations on the initial condition which makes the trajectory chaotic (however, we know that if the perturbation is small, this trajectory will be sticking KAM tori for a long time, it will need very long time to reach the density measure). Most trajectories in this chaotic region are Poincaré recurrent but not Bohr recurrent. Since the density measure of the chaotic trajectories cannot overlap those of the KAM tori, this density measure at infinite time cannot be the standard Gibbs measure (which corresponds to a uniform distribution in the submaniflod at constant energy and norm). This situation can be only obtained when the measure of KAM tori is negligible and then most chaotic trajectories should behave statistically like a thermalized state (if any).

For infinite systems with initial conditions which are extensive (that is, with an infinite energy with nonvanishing density and norm), the norm is infinite and then our arguments agree with the general belief that the probability to generate KAM tori is likely zero. Consequently, thermalization is expected to occur at large time for most initial conditions and at a nonvanishing temperature. Note that a standard thermalized state might not exist for initial conditions with large norm compared to energy [Rasmussen *et al.*, 2000].

When the energy of the initial condition of the wave-packet in the infinite system, is finite (nonextensive), its energy density is zero. Then, if thermalization occurs at infinite time, the temperature of the system should be zero.This would be the situation in case of complete spreading. However, real thermalization could occur only on condition the KAM tori have negligible probability to occur but we have seen above that this might not be true. We expect this situation when the measure of KAM tori is really negligible that is when the linear spectrum at infinite size has an absolutely continuous spectrum (for initial conditions without self-trapping). The behavior should be quite different for finite energy trajectories in an infinite nonlinear system with purely discrete linear spectrum.

In early works, the possible existence of KAM tori was discarded, and then it was assumed that the motions in the core of the wave-packet was chaotic with a broadband Fourier spectrum (for most initial conditions). Thus, the core of the wave-packet should excite the nearby Anderson modes by a broadband force, then spread the energy of the wave-packet. However, while the wave-packet spreads, its amplitude decays so that the nonlinear interaction also decays. Chaos which is essentially due to nonlinearity also decays (it could be measured by the largest Lyapunov exponent which decays as a function of time). This effect is considered as the origin of the slowing down of the diffusion yielding the so-called subdiffusive behavior where the second moment of the energy or norm distribution diverges as t^{α}. The exponent α is smaller than its value for normal diffusion 1 [Shepelyansky, 1993; Molina, 1998; Pikovsky & Shepelyansky, 2008; Skokos *et al.*, 2009]. It roughly ranges about 1/3 but it is not yet known if its value is really universal. The argument for subdiffusion assumes that broadband chaos persists forever and discards the possibility of self-organization of this chaos that was suggested in Sec. 5.3. After infinitely long time, the wave-packet may converge to limit where chaos becomes a narrow band (singular continuous or almost periodic) so that diffusion stops. However, this process of convergence might be extremely long and beyond current computer ability.

We define very generally the limit set of a trajectory $X(t)$ in the phase space as the set of trajectories in $\bigcap_T \overline{\bigcup_{t\geq T} X(t)}$. For a system with infinitely many dimensions, the closure is considered for the weak (or product) topology. This limit set is not an attractor in Hamiltonian systems (which are nondissipative). Indeed it cannot be in finite size systems because the volume in the phase space has to be conserved. Actually for most chaotic trajectories, the limit set is the closure of the chaotic region and when it can be assumed the Arnol'd conjecture holds, it should be the whole phase space. For the KAM trajectories, this limit set is nothing but the KAM torus itself.

In infinite size systems, if there is complete spreading this limit state should be the single point $X(+\infty) = \{0\}$. If the limit state is not zero, this limit state is said to be nontrivial. It could depend on the initial condition and moreover might be highly sensitive to it when there is initial chaos.

We are going to present some arguments supporting the conjecture that the limit state may not be zero at least in some cases.

5.1. *Proof of self-trapping by norm conservation*

We first prove that in DNLS-like models, initial chaos might not imply complete spreading. Actually we prove that at least a part of the wave-packet must remain well focused.

DNLS models have the property to conserve both energy and norm. We take advantage of this for proving that norm conservation may forbid complete spreading for initial wave-packets which have a large enough norm. We mean by complete spreading that $\lim_{t\to+\infty} \sup_n |\psi_n(t)|^2 = 0$. Otherwise big packets would survive forever despite the fact that they could themselves be randomly moving and generating a kind of diffusion.

Generalized DNLS models are described by Hamiltonians $H(\{\psi_n\},\{\psi_n^\star\})$ which are functions of complex variables ψ_n and their conjugate variables $i\psi_n^\star$ invariant by global and arbitrary phase rotation θ $\psi_n \to \psi_n e^{i\theta}\psi_n$ and $\psi_n^\star \to \psi_n e^{-i\theta}\psi_n^\star$. This implies that the dynamical equation

$$i\dot{\psi}_n = \frac{\partial H}{\partial \psi_n^\star} \tag{20}$$

conserve the total norm $\mathcal{N}^2 = \sum_n |\psi_n|^2$.

We consider a Hamiltonian on an arbitrary lattice $\mathbb{Z}^d$ which can be decomposed into its quadratic part $H_L = \langle\Psi|A_L|\Psi\rangle$ (where A_L is a linear self-adjoint operator) and a higher order nonlinear term H_{NL}

$$H = H_L + H_{NL}. \tag{21}$$

Operator A_L on a discrete lattice is chosen with a bounded spectrum both from above by ω_M and from below by ω_m (for example, A_L involves only short range interactions). Thus, we have for the linear part of the energy

$$\begin{aligned}\omega_m\langle\Psi\mid\Psi\rangle = \omega_m\mathcal{N} &\leq H_L(\{\psi_n\},\{\psi_n^\star\})\\ &\leq \omega_M\langle\Psi\mid\Psi\rangle = \omega_M\mathcal{N}.\end{aligned} \tag{22}$$

By definition, the nonlinear part of the energy H_{NL} is higher order than quadratic. As a consequence, $H_{NL}(\{\psi_n\},\{\psi_n^\star\})$ goes to zero if $\sup_n|\psi_n|^2$ goes to zero. This property is indeed well known in the limit of small amplitude where most energy is quadratic so that the linear approximation is valid. Otherwise this result is straightforward to prove. Let us consider for simplicity the example $H_{NL} = \sum_n |\psi_n|^4$. Then we have

$$\begin{aligned}H_{NL} = \sum_n |\psi_n|^4 &\leq \left(\sup_n |\psi_n|^2\right)\sum_n |\psi_n|^2\\ &= \left(\sup_n |\psi_n|^2\right)\mathcal{N}^2.\end{aligned}$$

Since the norm is conserved, H_{NL} goes to zero if $\sup_n|\psi_n|^2$ goes to zero, which proves our assertion. This proof can be easily extended to more complex forms of H_{NL}.

If the wave-packet spreads to zero ($\lim_{t\to+\infty} \times \sup_n|\psi_n(t)|^2 = 0$), the nonlinear part of energy $H_{NL}(\{\psi_n(t)\},\{\psi_n^\star(t)\})$ vanishes at $t = +\infty$. Since the total energy H is conserved, we have at any time $H = H_L + H_{NL}$ and consequently $H = H_L(\{\psi_n(+\infty)\},\{\psi_n^\star(+\infty)\})$. Then because of Eq. (22), we should have

$$\omega_m\mathcal{N} \leq H \leq \omega_M\mathcal{N}. \tag{23}$$

For proving that there exists wave-packets which cannot spread, we have to chose an initial wave-packet with large enough amplitude so that in Eq. (23) is not fulfilled. This can be easily done when the nonlinearity is homogeneous with a higher order $n > 2$ than quadratic. Then, considering an initial wave-packet $\{\lambda\psi_n(0)\}$ with amplitude parameter λ, its norm is proportional to λ^2 as well as its linear energy H_L, while its nonlinear energy H_{NL} is proportional to λ^n. As a result, the ratio of the total energy with the norm is proportional to λ^{n-2} which diverges for large λ. Consequently, if we choose λ large enough, in order that Eq. (23) is not fulfilled, when choosing such an initial condition $\{\lambda\psi_n(0)\}$ for the wave-packet, it cannot spread to zero and consequently a part of the initial energy must self-trap. This situation is characterized by the fact that the participation ratio of the wave-packet $P(t) = \mathcal{N}^2/(\sum_n |\psi_n(t)|^4)$ remains bounded at all time. Note that in contrast, the second moment of the norm distribution $M_2(t) = \sum_n n^2|\psi_n(t)|^2$ (or of energy distribution) may diverge. When the nonlinear energy is not homogeneous, the same result may hold if we can still prove that in Eq. (23) is not fulfilled at enough large amplitude λ but this should depend on the specific form of the nonlinearity.

For example, in the DNLS model Eq. (3) where $H_L = \sum_n \epsilon_n|\psi_n|^2 - C(\psi_{n+1}\psi_n^\star + (\psi_{n+1}^\star\psi_n))$ and $H_{NL} = (\chi/2)\sum_n|\psi_n|^4$, we have for a single site initial wave-packet $H_{NL} = (\chi/2)\mathcal{N}^2$. Thus we obtain that complete spreading is impossible when at the initially excited single site n_0 we have $|\psi_{n_0}(0)| > \sqrt{(2\Omega_M/\chi)}$ (for $\chi > 0$) where Ω_M is the sup of the linear frequencies.

Let us remark that randomness does not play any role in our argument, which holds both for random or nonrandom Hamiltonians with spatial periodicity. However, the long time behavior of the wave-packet should not be the same and depends on whether the linear spectrum is purely discrete or contains an absolutely continuous part.

In the case where the linear spectrum has an absolutely continuous part (as in spatially periodic systems) and thus could be radiating, we expect that a part of the energy remains localized either as DBs (or possibly as two frequencies of quasiperiodic DBs [Johansson & Aubry, 1997]) since according to our theory, they are the only possible localized solutions sustained by the system and thus are the only possible candidates as limit states. In that case, the rest of the energy should spread at infinity through quasilinear radiations. This was already the situation observed in the pioneering paper [Sievers & Tekeno, 1988] despite not being a DNLS model.

The situation should be quite different when the linear spectrum is purely discrete since as shown above, there are many other localized solutions which are almost periodic and are candidates of limit solutions as well as DBs, but there is no clear numerical evidence that one of them could be the limit of the wave-packet at infinite time.

Finally, let us note that our proof of self-trapping does not prove that the limit state of the wave-packet cannot be zero because the wave-packet might remain well focused but move at infinity, for example, in a randomly diffusive way. However, up to now such a behavior has never been observed in numerics.

5.2. *A nonlinear model with no spreading*

If we choose the linear operator A_L proportional to the identity, the linear spectrum is fully degenerate and despite no disorder in this linear operator, there is no linear radiation. We obtain a class of DNLS models where the linear energy is proportional to the norm $H_L = \Omega\mathcal{N}^2$ and thus is time constant. Since the total energy $H_L + H_{NL}$ is time constant, the nonlinear energy H_{NL} is also time constant. Consequently, if an initially localized wave-packet would spread to zero, this nonlinear energy should go to zero, which is impossible since it is time constant. Consequently, any wave-packet cannot spread to zero in that class of model. However, our argument does not tell what is the long time behavior of this wave-packet. For example, despite remaining well focused, it could move uniformly or randomly.

As an example of models where wave-packet spreading is impossible, and where it would be interesting to study numerically wave-packet behavior, let us propose the Hamiltonian

$$H = \sum_n \epsilon_0|\psi_n|^2 + \chi_n|\psi_n|^4 + C_n|\psi_{n+1} - \psi_n|^4 \tag{24}$$

ϵ_0 is a constant. The quartic coupling is purely nonlinear. Coefficients χ_n and C_n may be chosen random or constant.

Models where the linear spectrum is degenerate at a single value were studied earlier as *compacton models*. A similar model with no linear dispersion was studied earlier [Maniadis & Bountis, 2006] in a spatially periodic model with no disorder. However, their model does not belong to the class of DNLS model and thus has no norm conservation. It was found numerically in that model, that there exist quasiperiodic solutions (KAM tori) in wide regions near DBs and also there exist broadband chaotic DBs. Note that the result of [Aubry & Schilling, 2009] which forbids the existence of broadband chaotic solutions does not hold in that model. This conclusion was obtained assuming a purely discrete linear spectrum which is dense in intervals but it does not hold in principle in models where the linear spectrum is degenerate at a single value.

5.3. *Limit state or spreading? Conjectures*

As shown above, there are initial single site wave-packets which yield almost periodic stationary solutions and do not spread at all. Many such wave-packets are found in wide regions close to the anticontinuous limit (strong disorder i.e. short localization length) while their domain of existence shrinks to small norms while the localization length grows.

If we choose an initial wave-packet in the large gaps where KAM tori are absent, chaos is initially very strong and spreading immediately manifests. This was the situation in most early numerical experiments showing spreading. However, as the wave-packet spreads its maximum amplitude become smaller and smaller, so that after a certain time the wave-packet necessarily enters a region where according to our empirical bound Eq. (16), there are KAM tori at the same norm and comparable amplitude which exist with nonvanishing probability. Since KAM tori are stationary time reversible solutions, our initially chaotic wave-packet cannot generate by itself such a KAM torus as we have seen above but it may approach very close to some of them. The sticking of these KAM tori for long time could be manifest as a kind of bottleneck for the diffusion process associated to drastic slowing down of the diffusion.

We may expect this bottleneck effect by using the results on the statistics of resonances as in [Krimer & Flach, 2010]. Krimer and Flach found within the assumption of a rather uniform norm distribution in the localization volume that an Anderson mode ν becomes nonresonant when its norm square n becomes sufficiently small, that is smaller than some m_ν which depends on the disorder realization. The probability density $Q(m)dm$ of the distribution of m_ν over all disorder realization yields the probability density $P(n) = \int_0^n Q(m)dm$ that $m \leq n$ in order that a given Anderson mode ν with square norm n is resonant. Since it was found $P(n) \sim 1 - e^{-\chi nc}$, we also have $Q(m) \sim e^{-\chi nc}$ since it is the derivative of $P(n)$ with respect to n.

Thus for a given disorder realization, we can associate positive numbers m_ν to each Anderson mode which correspond to the maximum norm square it can sustain before being involved in a resonance. Each of these numbers depends randomly on the local realization of the disorder within the localization volume. They are uncorrelated beyond this localization volume (since the disorder is uncorrelated) and their probability density is $Q(m)dm$.

Let us now assume that for a given disorder realization and after some time, the single site wave-packet with square norm $\mathcal{N}^2$ is initially chaotic and has spread rather uniformly over p sites where p is much larger than the localization volume. We mean by rather uniformly the fact that the norm distribution inside the core of the wave-packet is kept rather compact, without important fluctuations (this fact was checked numerically by measuring the so-called compactness index). Particularly, this means that the wave-packet does not break into distant smaller wave-packets with relatively high peaks. Then, the square norm n for each excited Anderson mode is roughly of order $\mathcal{N}^2/p$. Consequently, each Anderson mode ν which is resonant up to some time, stops to be resonant after the spreading p is large enough. The probability that all excited modes in the wave-packet are nonresonant is roughly $\sim \prod_{\nu=1}^{p} e^{-\chi c \mathcal{N}^2/p} = e^{-\chi c \mathcal{N}^2}$. This probability is nonvanishing, consequently, if the wave-packet spreads uniformly it will necessarily happen at some time and will not sustain any resonance. Would it mean that it becomes a KAM torus? This is impossible because the wave-packet should have been stationary from the beginning of the evolution and thus should not spread substantially beyond the localization volume. Does it converge to a single KAM torus? If it does, this process should be very slow because the Lyapunov exponents of KAM tori are zero.

We propose another scenario we believe to be more realistic than this one. We note that the above resonance argument takes only into account the lowest order resonances. Actually, there are many higher order resonances which should also generate chaotic Arnol'd webs between the KAM tori (which actually form a fat Cantor set). Those webs become thinner and thinner and their chaos weaker and weaker as the resonance order increases. It seems more plausible that in the absence of strong resonances, the trajectory of the wave-packet would nevertheless remain chaotic by evolving randomly along such thin and weakly chaotic Arnol'd webs. The Lyapunov exponents of the trajectory measured over this transient state should drop. We then expect that the wave-packet continues to spread but at a much slower rate. This long lasting regime of slow diffusion corresponds to a dynamical bottleneck. When spreading sufficiently more, the wave-packet should reach again an extension beyond which the wave-packet may find another low order resonance with the same finite probability as $1 - e^{-\chi c \mathcal{N}^2}$. At a later time, it should meet another bottleneck and so on. It would appear the trajectory of the wave-packet intermittently should be sticking KAM tori closer and closer and for longer and longer times as spreading increases.

The careful observation of the behavior of the log–log plot of second moment of a spreading wave-packet for a single disorder realization [Pikovsky & Shepelyansky, 2008; Flach, 2008], shows that the second moment growth measuring the wave-packet spreading seems to exhibit few accidents after long time evolutions. The growth slows down during transient times which becomes longer and longer as the time increases. Further investigation is necessary to check if they can be viewed as KAM bottlenecks as we propose. Detailed investigation of the Lyapunov exponents and of its transient fluctuations versus time would be very useful. However, let us also remark that the role of the numerical errors may become very crucial especially when passing bottlenecks. These numerical errors act as small numerical noise negligible in the case of strong chaos, by which it might become the main cause of diffusion helping pass faster narrow KAM bottlenecks where chaos is weaker.

It is not yet clear if these KAM bottlenecks will forbid complete spreading at infinite time. A possible scenario which would be compatible with the fact that complete spreading is impossible in some cases, is that the diffusion through the many KAM bottlenecks may not be sufficient to allow complete spreading. Only a part of the energy of the core could pass these bottlenecks. Then, the core of the wave-packet may approach almost stationary KAM tori while weak chaos survive mostly in the tail of the wave-packet, which thus continue to slowly extend through the infinite system. Thus we could have a long tail limit profile with infinite second moment while the participation ratio remains bounded. Such limit profile for an initially localized wave-packet with long tail are known [Lepri *et al.*, 2010], though it is for different model with acoustic modes and with disorder but linear. Another argument for a limit profile with long tail is that having a short tail would be in some sense equivalent to diffusion in a finite size system. But this would contradict the Arnol'd conjecture described above which we believe to be valid (only) for finite size systems.

Finally, we conjecture that the limit profile solution if it exists, could be marginally chaotic with a Fourier spectrum which is not broadband but singular continuous. This phenomena can be viewed as a self-organization of the initial chaos in order to remove all possible radiations at infinity and thus to stop spreading. As was suggested in [Aubry & Schilling, 2009], in simplified models, it is in principle possible that such solutions exist without radiation through the linear purely discrete spectrum. We suggest these limit solutions could correspond to KAM solutions at criticality which lies precisely at the border of the gaps of resonances (delimiting the Arnol'd webs). Similar solution was described long ago in finite size nonconservative systems [Grebogi *et al.*, 1984; Yalçinkaya & Lai, 1996]. They are still chaotic but only marginally chaotic with zero Lyapounov exponent.

We suggest, for easier progress on that problem in the future, that instead of investigating random DNLS models, we investigate first both numerically and analytically, the diffusion of a wave-packet in nonrandom DNLS models with a purely discrete linear spectrum, for example those with a sine quasi-periodic potential [Aubry & André, 1980]. The advantage of this approach would be (i) To avoid statistics on many disorder realizations and thus save a huge amount of computing time. (ii) It might be possible by choosing an incommensurability ratio such as the golden mean, to construct good commensurate approximations of the system where the Fibonacci numbers are space periods and in which the long time behavior of a given initial wave-packet can be investigated. In these periodic systems where the linear spectrum exhibits many bands and gaps, wave-packet diffusion does exist in principle but note also that because of the nonlinearity, the spontaneous formation of DBs and also gap DBs may trap substantial parts of the energy. Numerical investigations with a scaling analysis interpretation could be attempted for reaching informations about the fully incommensurate system. It should also be possible to investigate the KAM tori in a series of finite size systems with Fibonacci numbers where they do exist and again to attempt to confirm the existence of KAM tori in the infinite incommensurate system by scaling analysis.

6. Concluding Remarks

We have not proven that complete subdiffusion of an initially localized wave-packet is always impossible, but we have shown numerically that initial wave-packets chosen in appropriate range of parameters do not exhibit any subdiffusion over computing times comparable to those used early and where subdiffusion was claimed. Those wave-packets correspond to stationary KAM tori which are almost periodic in time and we gave arguments for their existence as exact solutions.

For the other wave-packets which clearly exhibit initially a chaotic behavior, we have proven that there are initial conditions where in spite of this observation, subdiffusion must either be incomplete or at worst is totally impossible but for special models only. We suggested a speculative scenario of spreading in order to be compatible both with the numerical observations and the few available rigorous results. After sufficiently long time, subdiffusion is disturbed by the so-called KAM bottlenecks forcing diffusion to operate in thinner Arnol'd chaotic webs.

There is still a very difficult unsolved question. Is it really possible to obtain complete spreading (and subdiffusion) at infinite time, at least in some cases or models? Or is there self-organization of the wave-packet as a limit profile? If so what are the properties of this limit profile, long tail, sensitivity to initial conditions?... Numerical tests are clearly limited by the actual computer efficiency, since very long computing time with very high accuracy are both required. Obtaining an answer to this question seems to become a mathematical problem, at least for finding new and original investigation methods for the numerics. We hope that these studies will stimulate further attempts towards more rigorous treatments, as well as more refined numerical studies.

Acknowledgments

The MPIPKS Dresden is warmly acknowledged for its support by many invitations since 2007 and for the Gutzwiller fellowship 2009–2010. The author specially thanks Sergej Flach for very fruitful discussions during that period and also, George Kopidakis and Magnus Johansson who helped very much to develop the new ideas presented in this paper.

References

Albanese, C. & Fröhlich, J. [1991] "Perturbation theory for periodic orbits in a class of infinite dimensional Hamiltonian systems," *Commun. Math. Phys.* **138**, 193.

Arnol'd, V. I. [1964] "Instability of dynamical systems with several degrees of freedom," *Sov. Math. Dokl.* **5**, 581–585.

Aubry, S. & André, G. [1980] "Analyticity breaking and Anderson localization in incommensurate lattices," *Ann. Israel Phys. Soc.* **3**, 133–164.

Aubry, S. [1997] "Breathers in nonlinear lattices: Existence, linear stability and quantization," *Physica D* **103**, 201–250.

Aubry, S., Kopidakis, G. & Kadelburg, V. [2001] "Variational proof for hard discrete breathers in some classes of Hamiltonian dynamical systems," *DCDS-B* **1**, 271–298.

Aubry, S. [2006] "Localization and transfer of energy in discrete nonlinear systems. Prospective applications," *Physica D* **216**, 1–30.

Aubry, S. & Schilling, R. [2009] "Anomalous thermostat and intraband discrete breathers," *Physica D* **238**, 2045.

Bredikhina, E. A. [2002] "Almost-periodic function," *Encyclopaedia of Mathematics*, ed. Hazewinkel, M. (Springer, Berlin).

Bourgain, J. & Wang, W.-M. [2008] "Quasi-periodic solutions of nonlinear random Schrödinger equations," *J. Eur. Math. Soc.* **10**, 1.

Contopoulos, G., Voglis, N., Efthymiopoulos, C., Froeschle, C., Gonczi, R., Lega, E., Dvorak, R. & Lohinger, E. [1997] "Transition spectra of dynamical systems," *Cel. Mech. Dyn. Astron.* **67**, 293.

Contopoulos, G. & Harsoula, M. [2010] "Stickiness effects in conservative systems," *Int. J. Bifurcation and Chaos* **20**, 2005–2043.

Fishman, S., Krivolapov, Y. & Soffer, A. [2009] "Perturbation theory for the nonlinear Schrodinger equation with a random potential," *Nonlinearity* **22**, 2861.

Flach, S. & Gorbach, A. V. [2008] "Discrete breathers — Advances in theory and applications," *Phys. Rep. — Rev. Sec. Phys. Lett.* **467**, 1–116.

Flach, S., Krimer, D. O. & Skokos, Ch. [2009] "Universal spreading of wave-packets in disordered nonlinear systems," *Phys. Rev. Lett.* **102**, 024101; *ibid* [2009] **102**, 209903.

Froeschlé, C. & Scheidecker, J.-P. [1975] "Stochasticity of dynamical systems with increasing number of degrees of freedom," *Phys. Rev. A* **12**, 2137.

Froeschlé, C. [1978] "Connectance of dynamical systems with increasing number of degrees of freedom," *ibid. A* **18**, 277.

Fröhlich, J., Spencer, T. & Wayne, C. E. [1986] "Localization in disordered, nonlinear dynamic-systems," *J. Stat. Phys.* **42**, 247.

Gignoux, F. & Silvestre-Brac, B. [2009] *Solved Problems in Lagrangian and Hamiltonian Mechanics* (Springer, Dordrecht).

Grebogi, C., Ott, E., Pelikan, S. & York, J. A. [1984] "Strange attactor that are not chaotic," *Physica D* **13**, 261–268.

Johansson, M. & Aubry, S. [1997] "Existence and stability of quasiperiodic breathers in the discrete nonlinear Schrödinger equation," *Nonlinearity* **10**, 1151–1178.

Johansson, M. & Aubry, S. [2000] "Growth and decay of discrete Schröedinger breathers interacting with internal modes or standing-wave phonons," *Phys. Rev. E* **61**, 5864–5879.

Johansson, J., Kopidakis, G. & Aubry, S. [2010] "KAM tori in 1D random DNLS models?" *EPL* **91**, 50001.

Kopidakis, G. & Aubry, S. [2000] "Discrete breathers and delocalization in disordered nonlinear systems," *Phys. Rev. Lett.* **84**, 3236; *ibid.* [2000] "Intraband discrete breathers in disordered nonlinear systems II: Localization," *Physica D* **139**, 247.

Kopidakis, G., Komineas, S., Flach, S. & Aubry, S. [2008] "Absence of wave-packet diffusion in disordered nonlinear systems," *Phys. Rev. Lett.* **100**, 084103.

Krimer, D. & Flach, S. [2010] "Statistics of wave interactions in nonlinear disordered systems," *Phys. Rev. E* **82**, 046221.

Lahini, Y., Avidan, A., Pozzi, F., Sorel, M., Morandotti, R., Christodoulides, D. N. & Silberberg, Y. [2008] "Anderson localization and nonlinearity in one-dimensional disordered photonic lattices," *Phys. Rev. Lett.* **100**, 013906.

Lepri, S., Schilling, R. & Aubry, S. [2010] "Asymptotic energy profile of a wave-packet in disordered chains," *Phys. Rev. E* **82**, 056602.

MacKay, R. S. & Aubry, S. [1994] "Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators," *Nonlinearity* **7**, 1623.

Maniadis, P. & Bountis, T. [2006] "Quasiperiodic and chaotic discrete breathers in a parametrically driven system without linear dispersion," *Phys. Rev. E* **73**, 046211.

Molina, M. I. [1998] "Transport of localized and extended excitations in a nonlinear Anderson model," *Phys. Rev. B* **58**, 12547.

Pikovsky, A. S. & Shepelyansky, D. L. [2008] "Destruction of Anderson localization by a weak nonlinearity," *Phys. Rev. Lett.* **100**, 094101.

Pöschel, J. [1990] "Small divisors with spatial structure in infinite dimensional Hamiltonian systems," *Commun. Math. Phys.* **127**, 351.

Rasmussen, K. Ø., Cretegny, T., Kevrekidis, P. G. & Grnbech-Jensen, N. [2000] "Statistical mechanics of a discrete nonlinear system," *Phys. Rev. Lett.* **84**, 3740.

Shepelyansky, D. L. [1993] "Delocalization of quantum chaos by weak nonlinearity," *Phys. Rev. Lett.* **70**, 1787.

Sievers, A. J. & Takeno, S. [1988] "Intrinsic localized modes in anharmonic lattices," *Phys. Rev. Lett.* **61**, 970.

Skokos, Ch., Krimer, D. O., Komineas, S., Flach, S. *et al.* [2009] "Delocalization of wave-packets in disordered nonlinear chains," *Phys. Rev. E* **79**, 056211.

Wang, W.-M. & Zhang, Z. [2009] "Long time Anderson localization in the nonlinear random Schrödinger equation," *J. Stat. Phys.* **134**, 953.

Yalçinkaya, T. & Lai, Y. C. [1996] "Blowout bifurcation route to strange nonchaotic attractors," *Phys. Rev.* **77**, 5039–5042.